Time and the Evolution of States in Relativistic Classical and Quantum Mechanics*

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Abstract: A consistent classical and quantum relativistic mechanics can be constructed if Einstein's covariant time is considered as a dynamical variable. The evolution of a system is then parametrized by a universal invariant identified with Newton's time. This theory, originating in the work of Stueckelberg in 1941, contains many questions of interpretation, reaching deeply into the notions of time, localizability, and causality. Some of the basic ideas are discussed here, and as an example, the solution of the two body problem with invariant action-at-a-distance potential is given. A proper generalization of the Maxwell theory of electromagnetic interaction, implied by the Stueckelberg-Schrödinger dynamical evolution equation and its physical implications are also discussed. It is also shown that a similar construction occurs, applying similar ideas, in the theory of gravitation.

1. Introduction

One of the most profound and difficult problems of theoretical physics in this century has been the construction of a simple, well-defined theory which unites the ideas of quantum mechanics and relativity. In fact, such a problem has existed in the consistent formulation of classical relativistic dynamics as well. A central problem in formulating such a theory is posed by the description of the state of a system which has *spatial* properties (we shall come later to a discussion of spin).

Nonrelativistic quantum mechanics, making explicit use of the Newtonian notion of a universal, absolute time, provides a description of such a state in terms of a square integrable function over spatial variables at a given moment of time. This function is supposed to develop dynamically, from one moment of time to another, according to Schrödinger's equation, with some model Hamiltonian operator for the system.

This description of a state is inconsistent with special relativity from both mathematical and physical points of view. The wave function described in a frame in motion with respect to the frame in which the state is originally defined takes on the meaning of a probability amplitude at a set of different times, depending on the value of the spatial variables. Since the Hilbert spaces associated with different times are distinct, it therefore loses its interpretation as the description of a state.

The situation for classical nonrelativistic mechanics is quite analogous; the state of a system is described by a set of canonical coordinates and momenta (the variables of the phase space) at a given time. These canonical variables develop in time according to the first order Hamilton equations of motion.

The variables of the phase space, under the transformations of special relativity, are mapped into a new set in which the time parameter for each of them depends on the spatial location of the points; in addition, there is a structural lack of covariance of the phase space variables themselves. The restoration of the original description by extrapolating the dynamical evolution through this family of times is not satisfactory since, in addition to being highly impractical, the specification of a state should be independent of the model for the dynamical evolution of a system.

On the other hand, observed interference phenomena, such as the Davisson-Germer experiment, showing the interference pattern due to the coherence of the wave function over the spatial variables at a given time, clearly should remain when observed from a moving frame (detectors in motion relative to the original experiment). Hence one would expect that there is a more general, covariant, description of the state of a system, which would predict such an interference pattern, modified only by the laws of special relativity.

In the following, I describe the fundamental problem of localization introduced by the use of the usual relativistic wave equations [1]. I then turn to a discussion of Einstein's notion of the (covariant) time, and show that it is possible to construct a manifestly covariant classical and quantum theory in a framework, first proposed in a complete form (for a single particle) by Stueckelberg [2] in 1941, and, much later, developed for systems with more than one particle [3], in which localization is rigorous. This framework involves treating all four components of energy-momentum as independent variables, reflecting the understanding of the Einstein time as a non-trivially measureable quantity, and does not restrict a particle to an infinitely sharp mass shell. One therefore considers *reducible*

representations of the Poincaré group. Some fundamental questions, such as achieving an understanding of the Newton-Wigner operator [4], and the Landau-Peierls uncertainty relation [5] are then discussed. As a concrete application, I discuss the solution of the two body problem. The extension of the theory to $U(1)$ gauge invariant form, introducing electromagnetic interaction [6], involves a proper generalization of the Maxwell theory which raises interesting questions for the structure of general relativity as well.

2. The problem of localization for the solutions of relativistic wave equations.

Attempts to take into account the required relativistic covariance of the quantum theory by means of relativistic wave equations such as the Klein-Gordon equation for spin zero particles, and the Dirac equation for spin 1/2 particles, have not succeeded in resolving the difficulties associated with the definition and evolution of quantum states. These equations are of manifestly covariant form, with the potential interpretation of providing a description of a quantum state, with spatial properties, in each frame, evolving according to the time parameter associated with that frame. Newton and Wigner [1], however, have shown that the solution $\phi(x)$, for example, of the Klein-Gordon equation, cannot have the interpretation of an amplitude for a local probability density. The function $\phi_0(x)$, corresponding to a particle localized at $\mathbf{x} = 0$, at $t = 0$, has support in a range of $\mathbf{x}$ of order $1/m$, where m is the mass of the particle. They found that the distribution corresponding to a localized particle is a (generalized) eigenfunction of the operator

$$\mathbf{x}_{NW} = i\left(\frac{\partial}{\partial \mathbf{p}} - \frac{\mathbf{p}}{2E^2}\right). \tag{2.1}$$

We shall return to discuss this problem, and its solution, in more detail later. A similar conclusion was found for the solutions of the spin 1/2 Dirac equation. The well-known problem posed by the lack of a positive definite probability density for the Klein-Gordon equation is formally managed by passing to the second quantized formalism; the Dirac equation admits a positive definite density, but the problem of localization remains. In both cases, in the second quantized formalism, the vacuum to one particle matrix element of the field operator, which should have a quantum mechanical interpretation (the one-particle sector), poses the same problem of localization. The prediction of the formation of interference patterns remains ambiguous in this framework.

Foldy-Wouthuysen [7] type transformations (for both spin zero and spin 1/2 cases) restore the local property of the wave functions [3], as for the non-relativistic Schrödinger theory, but in this representation, manifest Lorentz covariance is lost. It is clear that the problem of localization is a fundamental difficulty in realizing a covariant quantum theory by means of the usual wave equations.

Hegerfeldt [7] has shown, moreover, that a wave function defined at time $t = 0$ which is localized in the sense of compact support, or in the Newton-Wigner sense, or has fast fall-off in spatial directions, cannot maintain this degree of localization at time $t > 0$ if the free evolution is governed by such wave equations (it necessarily evolves out of the light cone).

3 The Einstein notion of time.

The resolution of the problem of localization posed above lies in the formulation of a quantum theory for particles which are not precisely constrained to a pointwise mass-shell, i.e., the relativistic particle is not restricted, in its definition, to an irreducible representation of the Poincaré group as advocated by Wigner in his fundamental paper of 1939 [9]. The kinematic definition

$$E^2 = \mathbf{p}^2 + m^2 \tag{3.1}$$

is, of course, maintained, but the quantity m^2 is to be considered as a *dynamical variable*, with values determined by the interactions in the physical system [3]. If one wishes to construct a theory of particles, on a phenomenological level, which does not make direct use of the energy stored in the fields that surround them, it is necessary to admit this degree of freedom. For example, the electron is seen to have an effective mass, in the context of Lamb shift calculations [10], differing from its free value when it is bound in an atom, and the effective masses of nucleons bound in the nucleus differ from their free values. One sees immediately from the assumption of this added degree of freedom that, in the quantum theory, the operators corresponding to the mass and the position of the particle are incompatible. In this way, one can understand the foundation of the Newton-Wigner problem [3], as follows.

The scalar product of Klein-Gordon "wave functions" is given by

$$\int \frac{d^3\mathbf{p}}{2E} \phi^*(\mathbf{p})\phi(\mathbf{p}) = (\phi_1, \phi_2). \tag{3.2}$$

Newton and Wigner start by assuming that $\phi_0(\mathbf{p})$ is the wave function of a particle localized precisely at $\mathbf{x} = 0$; they then assume that the function $\phi_a(\mathbf{p}) = \exp i\mathbf{p} \cdot \mathbf{a}$ is orthogonal to ϕ_0 under the scalar product (3.2). The integral then corresponds to the Fourier transform of the function $|\phi_0|^2(\mathbf{p})/2E$; since it vanishes for all $\mathbf{a}$ not zero, the integrand must be constant. This implies that

$$\phi_0(\mathbf{p}) = C\sqrt{E}, \tag{3.3}$$

and therefore that the Fourier transform, which is the wave function in space, is not a $\delta-$function (as it would be if $E = $ const, as in the nonrelativistic limit), but a function which is spread out to the Compton wavelength of the particle. Hence these wave functions do not localize the particle. Another way to undertand this, in the framwork of the Stueckelberg theory, is that the variable $\mathbf{x}$ does not commute with m^2, as defined in (3.1), which explicitly contains the momentum variable, and hence the mass of a particle and its position are incompatible. In the discussion of the covariant relativistic quantum theory below, I explain how the Stueckelberg theory is indeed local in its description of the elementary objects of the theory, points in spacetime that might be called "events," and in a decomposition of the expectation value of the position $\mathbf{x}$ over all mass shells, one finds the Newton-Wigner operator (2.1) at each sharp mass value in the integral.

If, as I have argued above, we consider the mass of a particle as a dynamical variable, $\mathbf{p}$ and E must be considered as independent dynamical variables. The complementary variables, dual to these, are $\mathbf{x}$ and t, and we must conclude that these are also independent

dynamical variables. To understand the role of t as a dynamical variable, rather than a c-number parameter, we must examine carefully the Einstein notion of time.

To do this, let us examine the structure of the Lorentz transformation, restricting for now our discussion to one space and one time dimension,

$$\Delta t' = \frac{\Delta t - v\Delta x}{\sqrt{1 - v^2}}, \tag{3.4}$$

where we take c, the velocity of light in vacuum, to be unity. This expression, in Einstein's original interpretation, corresponds to the difference between two values of time recorded by detectors equipped with clocks in a frame F' moving at velocity v with respect to a frame F in which the two signals are emitted in an interval of time Δt, at two places separated by an interval Δx. One may think of the two events as associated with the same emitting object; in this case, it follows that the emitter has moved from x_1 to x_2 in the time $t_2 - t_1$. If the emitter is at rest in F, then both signals are sent from the same point, and Eq.(3.4) reduces to

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - v^2}} \equiv \frac{\Delta s}{\sqrt{1 - v^2}}, \tag{3.5}$$

where Δs is usually referred to as the proper time of the emitter in F.

It should be emphasized that the difference between $\Delta t'$ and Δs is not at all associated with the distance between the detectors in F' and the emitters in F. We may consider the detection as carried out by a set of receivers in F' filling, in the sense of Wheeler and Taylor [11], all of spacetime with synchronized clocks relatively at rest in F', and in uniform motion relative to the emitter in F. The detectors that lie in the immediate neighborhood of the emitter in F at t_1 and t_2 receive the signals with no retardation due to spatial separation; since all the detectors are synchronized, the two readings obtained in this way, generally in two different detectors of F' (due to the uniform motion of this frame), t'_1 and t'_2, can be treated as being detected by equivalent clocks in F', i.e., a global detection in the F' laboratory.

It is essential that the clocks associated with with the detectors in F' be precisely physically equivalent to the clocks associated with the emitters in F, and set to run at the same rate, so there can be a basis for the comparison between $\Delta t'$ and Δt. This assumption, that there exists a *standard clock* which can be found embedded in every frame independent of its (uniform) motion, is implicit in any discussion of special relativity. It is, furthermore, an essential postulate of general relativity as well.

As we know from general relativity, the apparent rate of a clock in a gravitational field is affected by a change in the metric tensor (the mechanism of the gravitational redshift). Local inertial frames, according to the equivalence principle, are characterized by motion in which they are freely falling (moving along a geodesic), so that the clocks imbedded in these frames are not subject to the influence of the gravitational field. This condition must be met for the inertial frames F and F' that we discussed earlier. Moreover, if the clocks that we consider have a varying self-energy caused by springs under tension or batteries with stored chemical energy, the rate of recording time of these clocks may be affected by the gravitational fields induced by the corresponding local concentration of energy density. The standard universal clocks that we visualize as imbedded in each inertial frame must

therefore be *ideal* clocks, in the sense that they contain no self-energy induced frequency shifts. In this case the proper time interval referred to in Eq.(3.5) recorded by an ideal detector at rest with respect to the emitter is $\delta s = \Delta\tau$, where we have designated the interval on the ideal standard inertial clocks as $\Delta\tau$.

The notion of time, as generated by an ideal standard inertial clock existing, in principle, in every inertial frame (locally, in free fall, in the presence of gravitational fields), coincides precisely with the universal time postulated by Newton [12], and in terms of which equations of motion for systems in interaction may be formulated. The formulation of a theory of mechanics with "action at a distance" interaction, of the form $V(x_1^\mu, x_2^\mu)$ would, in fact, be meaningless without the postulate of such a universal time, since the choice of x_1^μ and x_2^μ as two points along the respective world lines must be determined by some agreement about which two points are in interaction. This agreement, or correlation, is determined by the assumption of the existence of a universal parameter τ. I do not claim that it is possible to synchronize clocks in relative motion. The ideal clocks in any frame may be set to any arbitrary initial value. What we explicitly assume, as a postulate, is that *there exists a universal time by means of which dynamical interactions are correlated.*

The assumption of a single universal time τ of the world, as for the Galilean time in nonrelativistic dynamics, has far-reaching consequences. An interesting question, closely related to the twin paradox, immediately arises in general relativity. We have operationally defined the universal time parameter τ in correspondence with the reading on an ideal inertial clock. However, the integral of $d\tau = \sqrt{-g_{\mu\nu} dx^\mu dx_\nu}$ [we use the metric $(-, +, +, +)$ in the local flat space limit] along two different geodesics which start at a common point and later cross at another point in spacetime, give two different results, in general. Hence, one may argue that this operational definition is not consistent with a universal τ. This argument rests on the generally accepted idea that two points which coincide in (local) Minkowski space correspond to the same object [13]. It arises from our use of Minkowski space as a complete coordinatization of physical phenomena. We are, however, free to postulate that two events which overlap in spacetime, but have different values of the universal time τ, are *not* identically the same physical objects. The separation can be easily seen graphically by adding τ as an additional coordinate in configuration space. One then sees that the projection of the geodesics onto spacetime appear to cross when the two trajectories do not, in fact, coincide. Whether the two objects at τ_1, τ_2 over this intersection can be in nontrivial interaction depends upon assumptions about the form of the interaction (and the properties of the appropriate Green's functions [14]). In this picture the "twins" age in free fall in precisely the same way, by definition. The potential model we considered earlier corresponds to interaction only at equal τ. The twins are never faced with a paradox.

As an example of the type of interactions that may take place in the neighborhood of a locally flat region, the propagators for generalized (as we shall discuss later) electromagnetic fields in this framework may have spacelike components for domains in Minkowski space which are separated by distances greater than the τ difference [16]. For dynamical correlations to exist it is therefore necessary that the τ difference be less than the scale of the spacelike separation; for nearby spacelike neighbors, such interaction therefore requires almost equal τ origin of signal and dynamical impact, such as in the potential

models referred to above. I do not use the term transmission of "information" here, since the classical notion of information, as we shall see below, requires integration over τ; in this case the spacelike components of the Green's function vanish.

To conclude this section, let us consider the simple example of the gravitational red-shift and its dynamical implications. Suppose that an ammonia molecule on the planet Jupiter radiates (we adopt here, for the sake of illustration, a semiclassical picture) due to the motion of the nitrogen atom through the plane of the three hydrogen atoms. The interval Δt_J between oscillations is related to an inertial proper time interval $\Delta \tau_J$ by the generic formula

$$\Delta t' = \frac{\Delta \tau}{\sqrt{-g_{00}}}, \tag{3.6}$$

obtained from the equation

$$d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu \tag{3.7}$$

for the invariant interval of general relativity. Here, the differential dx^μ for a clock at rest on the surface of Jupiter is taken to be just dt_J , so that only the g_{00} term of the metric tensor $g_{\mu\nu}$ appears. Applying the same formula to the intervals on Earth, one then divides one by the other (for example, ref. [15]), to obtain

$$\frac{\Delta t_J}{\Delta t_E} = \sqrt{\frac{g_{00}^E}{g_{00}^J} \frac{\Delta \tau_J}{\Delta \tau_E}}; \tag{3.7}$$

assuming that $\frac{\Delta \tau_J}{\Delta \tau_E}$ is unity, one obtains a results that agrees very well with experiment. The experiment raises two important basic questions: a) By what physical mechanism can the ammonia atom control its oscillation frequency relative to the time of a freely falling clock? b) Why should the proper time interval associated with the oscillation of two clocks, one on Earth and the other on Jupiter, be the same, so that cancellation can take place? The ammonia atoms are not in free fall, bound to the surface of their respective planets, and there is no obvious mechanism for the radiation frequency to be related to the time of a freely falling clock. Furthermore, whatever mechanics governs the motion of the atom, it is not clear that the proper time interval for emission of one cycle of a wave should be the same on Jupiter and the Earth,

These questions can be simply answered by assuming that there is a universal world time which drives the motion of the ammonia clocks in both environments, and that the radiation which is emitted is affected by the gravitational field. The differential equations satisfied by the atomic motions would contain the independent variable τ corresponding to this universal evolution, just as Newton's equations contain the Newtonian time in the nonrelativistic theory of mechanics. In fact, one may postulate that all machines run according to this universal time. The freely falling clock of Einstein is also a machine, and since there are no forces acting on it, and it is ideal (no batteries, springs which can wind down, or any other sources of change in its self-energy), it may be set to read the universal time on its face. The two ammonia clocks therefore run by the same universal time, and the fact that these configurations are governed by oscillations according to this rate, the radiation is emitted at this basic frequency (governed by the mechanical system),

and modified by the acceleration intrinsic to the local gravitational field according to the redshift formula.

This example illustrates the necessity of postulating a universal time which is not a clock, but a time according to which the clocks and all other machines run. In fact, it is logically not satisfactory to define time according to clocks, for the simple reason that clocks must run according to the flow of time themselves.

4. Covariant Classical and Quantum Mechanics

Stueckelberg's original paper discussed the possibility of pair annihilation and production in *classical* relativistic mechanics. He first considered the motion of an event in spacetime tracing out a free worldline, clearly a straight line moving within the light cone, as for the kinematics of a freely moving massive particle. He then points out that interaction can bend this worldline, as might be generally expected, and then moves on to the possibility that the worldline can bend so much that it curves back in time, that is, after reaching a maximum position along time axis, it turns and heads backward in time. The interpetation of such a collection of events in the laboratory is that of two objects moving toward each other in the sequence of time signals generated in the detectors of the laboratory for the occurrence of events (e.g., with Geiger counters, or in a spark chamber) until they meet and annihilate. This is the classical possibility of pair ahhihilation. Stueckelberg then points out that the time t is no longer adequate, under these conditions, to serve as an independent variable, since the curve would then not be a function (two values of the space variable correspond to one of the time). He introduced a parameter along the trajectory, τ, which then serves as the effective independent variable. Due to the reversal of four momentum in space-time, one easily sees that the object moving backward in t would appear to have opposite charge if it were thought of as having positive energy, and going *forward* in t; it is therefore identified as the antiparticle. This interpretation of the antiparticle was used by Feynman [17] in his construction of the standard quantum electrodynamics from a perturbative point of view in 1949.

Stueckelberg then proceeded to construct a classical mechanics in spacetime by defining a function K on the phase space which satisfies the Hamilton equations

$$\frac{dx^\mu}{d\tau} = \frac{\partial K}{\partial p_\mu} \qquad \frac{dp^\mu}{d\tau} = \frac{\partial K}{\partial x_\mu}, \tag{4.1}$$

where the index μ takes on values 0, 1, 2, 3 corresponding to the physical coordinates of time and space. These relations can be derived from a variety of variational principles, as for the usual nonrelativistic mechanics, extended to four dimensions. For the free particle, one may choose

$$K_0 = \frac{p^\mu p_\mu}{2M}, \tag{4.2}$$

where M is a dimensional scale parameter, an intrinsic property of the particle. The equations of motion then become

$$\frac{dx^\mu}{d\tau} = \frac{p^\mu}{M}, \ \frac{dp^\mu}{d\tau} = 0 \tag{4.3}$$

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so that, dividing the space components by the time component in the first of these, one finds

$$\frac{d\mathbf{x}}{dt} = \frac{\mathbf{p}}{E}, \tag{4.4}$$

where $p^0 \equiv E$, precisely the Einstein relation for velocity induced by setting the observational frame into motion with velocity $-\mathbf{v}$ with respect to the particle. Note that from (4.3),

$$\frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau} = \frac{p^\mu p_\mu}{M^2}$$
$$= -(\frac{ds}{d\tau})^2, \tag{4.5}$$

from which we see that the proper time squared $(ds)^2 = -dx^\mu dx_\mu$ is equal to $(d\tau)^2$ only if $p^\mu p_\mu = -M^2$, the so-called "mass shell" condition. In case there are batteries or springs that can run down, K, which is $K_0 + V$, may still be constant, and then K_0 changes with the changing V. In the nonrelativistic limit of the theory, one finds that M may play the role of the Galilean target mass [18]. Piron and I [3] extended this construction to systems of arbitrary N events; the corresponding N worldlines correspond to N- particle systems. The generalized Hamiltonian for such a system might be chosen, for example as the many-body action-at-a-distance form

$$K = \sum_i \frac{p_i^\mu p_{\mu_i}}{2M_i} + V(x_1, \ldots, x_N). \tag{4.6}$$

Stueckelberg proceeded to construct the quantum theory associated with this symplectic mechanics; one assumes the covariant commutation relations

$$[x^\mu, p^\nu] = ig^{\mu\nu}, \tag{4.7}$$

and the (Stueckelberg-Schrödinger) equation

$$i\frac{\partial \psi_\tau(x)}{\partial \tau} = K\psi_\tau(x), \tag{4.8}$$

a Shrödinger-like equation first order in τ. Here, we have denoted by x the four dynamical variables x^μ. The wave function $\psi_\tau(x)$ belongs to a Hilbert space over R^4, and the wave function then falls off integrably in both space and time directions. The Fourier transform to energy-momentum representation,

$$\psi_\tau(p) = \int d^4x e^{ip^\mu x_\mu} \psi_\tau(x), \tag{4.8}$$

enables us to express the evolution of a free particle as

$$\psi_\tau(x) = \int d^4p \exp\{-i\frac{p^\mu p_\mu}{2M}\} e^{ip^\mu x_\mu} \psi_0(p), \tag{4.9}$$

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so that the stationary phase point moves, as in the standard Ehernfest relation, approximately with the relations (4.3), the classical motion of a spacetime event along the free particle world-line.

In the nonrelativistic Schrödinger theory, one may demand that the theory remain invariant in form under a $U(1)$ phase transformation, of the form $\psi \to e^{i\Lambda}\psi$. In the representation of (4.7) for which $p^\mu \to -i\partial/\partial x_\mu$, we see that in the equation (4.8), p^μ must be replaced by the covariant derivative $p^\mu - e_0 a^\mu$ and , morevover, the τ-derivative, $i\partial_\tau$ (where we use an abbreviated form for the derivative) must be replaced by $i\partial_\tau + e_0 a_5$. Each of the gauge compensation fields a_α, functions of both x and τ, then undergo a gauge transformation [6].

$$a_\alpha \to a_\alpha + \frac{1}{e_0}\partial_\alpha\Lambda, \tag{4.10}$$

thus restoring the equations to their original form.

To define dynamical equations governing the structure of the corresponding gauge invariant "field strengths"

$$f_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha, \tag{4.11}$$

we define the Lagrangian density

$$\mathcal{L} = \frac{1}{4}f_{\alpha\beta}f^{\alpha\beta}, \tag{4.12}$$

along with the Lagrangian density generating the Stueckelberg-Schrödinger equation as a field equation. The usual variational principle results in the additional field equation

$$\partial_\beta f^{\alpha\beta} = j^\alpha, \tag{4.13}$$

where $j^\alpha = \{j^\mu, \rho\}$ is the conserved *five*-current associated with the Stueckelberg-Schrödinger equation, derived in exactly the same way as the conserved current of the nonrelativistic Schrödinger equation; ρ is just $|\psi_\tau(x)|^2$, and j^μ is the antisymmetric bilinear form for $-i\partial^\mu - e_0 a_\mu$. The conservation of j^α is consistent with the antisymmetry of the field strengths $f^{\alpha\beta}$.

It is a remarkable fact that, as we have seen from the Schrödinger case in three dimensions and the Stueckelberg case in four, that the requirement of gauge invariance implies supplementing the manifold of the measure space by one additional dimension. A theory in which a "world" is described at a given moment of the evolution parameter is supplanted by a world in which the evolution parameter is incorporated, in the same second order as for the spatial (or spacetime) variables, providing the homogeneity that could support a higher symmetry. One may think of these gauge fields as correlation fucntions between the sequence of worlds parametrized on the foliation provided by the evolution.

This apparent higher symmetry is not really present when there is interaction with the matter fields, since the density ρ cannot transform with a current that contains derivatives, and, as with the Schrödinger Galilean theory, they break this symmetry locally. In this latter case, the symmetry of the free field equations for the homogeneous, noninteracting, field equations is O(3,1). Experimentally, it was found that this symmetry, the Lorentz symmetry of the world, is, in fact, more accurate than the previously assumed Galilean

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symmetry. The fields associated with the gauge invariant extension of the Stueckelberg theory appear to exhibit an O(3,2) or O(4,1) symmetry, depending on the choice of metric for the raising and lowering of the fifth (τ) index [6]. In the gauge for which $\partial_\alpha a^\alpha = 0$, analogous to the Lorentz gauge, the equations (4.13) become, in the absence of sources,

$$\partial_\alpha \partial^\alpha a_\beta = \pm \partial_\tau^2 - \partial_t^2 + \nabla^2 = 0. \tag{4.13}$$

Under Fourier transform with repect to τ, the first term becomes $\mp m^2$, where m corresponds to the mass associated with this component of the field. The O(3,2) choice of metric therefore corresponds to a field with non-physical mass (tachyonic), and may be thought of as intrinsic radiation; its limit onto the light cone, for $m \to 0$ corresponds to Maxwell radiation. The choice of the O(4,1) metric carries a real physical mass, and could be used to represent a physical vector meson, for example. The dynamical transiton from the O(3,2) to the O(4,1) metric may represent vector meson photoproduction.

Note that the usual Maxwell theory is properly contained in this generalized form, which we call pre-Maxwell. Eq. (4.13), for the $\{\mu\}$ components alone, may be written as

$$\partial_\nu f^{\mu\nu} + \partial_\tau f^{\mu 5} = j^\mu. \tag{4.14}$$

Integrating over τ from $-\infty$ to ∞, with appropriate asymptotic conditions, the second term vanishes. The current consevation law for the Stueckelberg current,

$$\partial_\alpha j^\alpha = 0, \tag{4.15}$$

under similar integration, becomes

$$\partial_\mu \int d\tau j^\mu = 0, \tag{4.16}$$

a result known to Stueckelberg [2]. Hence, one sees, with Stueckelberg, that $\int d\tau j^\mu$ is the Maxxwell four-conserved current, and occurs on the right hand side of the integrated version of (4.14). The τ-integral of the left hand side may therefore be identified with the Maxwell fields $F^{\mu\nu}$; these *zero modes* of the pre-Maxwell theory are just the Maxwell fields. The transmission of information, according to antennas and receivers as we know them, is associated primarily with the zero modes of the fields and currents.

Finally, since $\int d\tau a^\mu$ are the Maxwell vector potentials A^μ, we see that the dimension of a^μ is $1/L^2$, so that the "charge" e_0 must have dimension L. The Lagrangian density (4.12) requires an additional factor, say, λ, of dimension length. Moreover, the field equations (4.13) contain a current on the right hand side which is, in reality, proportional to e_0. The ratio e_0/λ then corresponds to the Maxwell charge e, and one understands the restriction to the neighborhood of the zero mode as a property of statistical correlations, in τ, of the field (last of ref. [6]). It has been shown that, in a path integral quantization of the corresponding field theory, going to the limit in which only a small neighborhood of the zero mode survives is equivalent to Pauli-Villars regularization[19].

5. Applications

In this section, I conclude by describing some applications of the theory outlined above to give concrete illustrations of how these constructions, based on the fundamental premise that the Einstein time t is a dynamical variable, may be manifested in known theoretical frameworks as well as in the world represented by laboratory experiments.

I first discuss the two important theoretical results of Newton and Wigner [4] and of Landau and Peierls [5] mentioned above. Startin witht he wave functions in the momentum representation defined in (4.8), we may ask for the quantum mechanical expectation value of

$$(\mathbf{x}_{op})|_{t=0} = i\frac{\partial}{\partial \mathbf{p}} - \frac{1}{2}\{t, \frac{\mathbf{p}}{E}\}. \tag{5.1}$$

The second term is the shift vt extrapolating an event occurring at $(t, \mathbf{x})$ back to the zero-time axis. The operator sought by Newton and Wigner [4] was the one for which the spectrum provides the values of $\mathbf{x}$ at $t = 0$. In the computation of the expectation value,

$$< \mathbf{x}_{op} >= \int d^4p\, \psi^*(p)\mathbf{x}_{op}\psi(p), \tag{5.2}$$

we may change variables using the definition $E = \sqrt{\mathbf{p}^2 + m^2}$, where m is a variable. Then, $d^4p = d^3\mathbf{p}\,dm^2/2E$. For each value of m^2, the measure then coincides with that of Eq. (3.2). However, the momentum derivative acting on the wave function, intended to differentiate only the first three arguments, would now act freely on all four. We must therefore subtract the contribution $(\partial E/\partial \mathbf{p})(\partial_E) = (\mathbf{p}/E)(\partial_E)$. This cancels precisely the contribution of this term from the anticommutator, leaving, on each mass shell, the operator (2.1).

Landau and Peierls [5] studied the problem of the existence of a measurement of the first kind on the spectrum of an operator with continuous spectrum. After a carefull heuristic analysis making use of the time-energy uncertainty relation, they arrived at the inequality

$$\Delta p \Delta t \geq \hbar/c \tag{5.3}$$

as an estimate. There was some criticism of this relation, since it was not derived in the same rigorous way that the position-momentum relation is obtained as a Schwartz inequality in the quantum theory. However, we may think of a Geiger counter which measures the time of occurrence of the passage of a particle at a point x_0, and suppose that some event occurs at a time t and place x different from x_0. If that event is part of a world line, then it may be extrapolated back by dividing the distance by the velocity, i.e., one can define in analogy to the Newton-Wigner operator of Eq. (5.1), the operator

$$(t_{op})|_{x=x_0} = -i\partial_E - \frac{1}{2}\{x - x_0, \frac{E}{p}\}. \tag{5.4}$$

A calculation similar to that of the Newton-Wigner case discussed above provides a formula for the effective time operator on each mass shell. The commutator of p with the operator t_{op} does not vanish due to the presence of the second term, and one finds precisely the

commutator necessary to achieve the Landau-Peierls relation (5.3) rigorously, through the Schwartz inequality.

The crucial point in Hegerfeldt's proof [8] that acausal behavior is associated with the Dirac and Klein-Gordon fields is that the analyticity implied by the Fourier transform of compactly supported functions is destroyed by the application of the evolution operator $\exp\{-i\sqrt{\mathbf{p}^2 + m^2}\}$. The operator K_0 is simply quadratic in momentum, and hence analytic, so it may not change the analytic character of compactly supported functions. However, one must take care that a boundedness requirement on the range of $p_\mu p^\mu$ does not change this property; it appears that tachyon components, or a balance between particles and antiparticles (with some implications for the theory of detection) may be required [20].

A very important application of the two-body form of the theory in illustrating the idea of the existence of a dynamical time variable is that of the relativistic two body bound state. As a model for an action-at-a-distance potential, let us use the form $V(\rho)$, where

$$\rho = \sqrt{(x_1 - x_2)^2 - (t_1 - t_2)^2},\tag{5.5}$$

assuming that the two events described dynamically by the theory are spacelike separated so that the square root is well defined. Separating variables in the equation (4.8) to total momentum $P^\mu = p_1^\mu + p_2^\mu$ and relative momentum $p^\mu = (M_2 p_1^\mu - M_1 p_2^\mu)/(M_1 + M_2)$, Eq. (4.8) can be written

$$i\frac{\partial \psi_\tau}{\partial \tau} = \frac{P^\mu P_\mu}{2M} + \frac{p^\mu p_\mu}{2m} + V(\rho),\tag{5.6}$$

where $M = M_1 + M_2$ and m is the reduced mass $M_1 M_2/(M_1 + M_2)$. The center of mass motion represented by the first term may be factored out by applying a phase to the wave function, and the remaining reduced motion problem solved as a stationary eigenvalue problem. If the support of the wave function is taken as the full spacelike region, one separates variables in the d'Alembertian according to the definitions

$$\begin{aligned}
t &= \rho \sinh \beta \\
x_1 &= \rho \cosh \beta \cos \theta \cos \phi \\
x_2 &= \rho \cosh \beta \cos \theta \sin \phi \\
x_3 &= \rho \cosh \beta \sin \theta.
\end{aligned}\tag{5.7}$$

Following the usual procedure, the variable occurring the least number of times, ϕ, is separated first, then θ, then β. leaving the final "radial" equation in ρ for last. The solutions of the ρ equation, yielding the spectrum, depend on the separation variable of the β equation, a quantum number that has no simple nonrelativistic interpretation. The spectrum, furthermore, for the important case $V(\rho) = -e^2/\rho$ (which has the nonrelativistic limit $-e^2/r$) turns out[21] to be of the form $1/(n + \frac{1}{2})^2$, and disagrees with the observed results for atomic spectra. As pointed out by Bargmann [22], however, it is not possible to construct a representation of the Lorentz group induced on functions with support in the full spacelike region. There are two subregions of the spacelike measure space which are, with translations along the z-axis, transitive under the subgroup O(2,1) of the Lorentz

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group O(3,1), which may be described as the interior and exterior regions obtained by constructing two planes tangent to the light cone at opposite sides. The two regions are described by the conditions

$$x_1^2 + x_2^2 \geq t^2$$
$$x_1^2 + x_2^2 \leq t^2. \tag{5.8}$$

Separation of variables can be carried out in both of these; the second yields no bound states, and incorporates tunneling through the light cone, while the first provides solutions which are bound states, with spectrum $(1/n^2)$, consistent with experiment. In fact the radial equation obtained in this way coincides exactly with that of the nonrelativistic Schrödinger theory, and hence every nonrelativistic central potential problem has a corresponding manifestly covariant form with the same spectrum [23]. The variables are defined in this case as

$$t = \rho \sinh \beta \sin \theta$$
$$x_1 = \rho \cosh \beta \sin \theta \cos \phi$$
$$x_2 = \rho \cosh \beta \sin \theta \sin \phi \tag{5.9}$$
$$x_3 = \rho \cos \theta.$$

Separation occurs in the sequence ϕ first, then β (the separation constant provides a large degeneracy), then θ, and then ρ. Hence, the radial equation contains the separation constant which is associated with θ, which carries, in this case, the meaning of the angular momentum in the nonrelativistic limit. These wave functions are then satisfactory to describe the physically observable properties of a relativistic quantum system. We remark that, returning to (4.8), we must recall that the center of mass momentum was factored out. Writing the operator K as a sum of the two terms,

$$K = \frac{P^\mu P_\mu}{2M} + K'_{rel}, \tag{5.10}$$

one can solve, in the center of mass frame, for which $\mathbf{P} = 0$, for the total energy E squared. It is this quantity which is actually measured in the laboratory. Taking the asymptotic value $-M/2$ for K (established by assuming that when $V \to 0$ the asymptotically separated particles are close to their mass shells), and solving (5.10) for E, one finds that the leading contribution for small excitations is just the eigenvalues K'_{rel}; the next terms constitute relativistic corrections.

The solutions found in our study of the bound state problem can also be applied to the solutions of the scattering problem [24], to the problem of calculating the selection rules and amplitudes for radiative decay [25] and to the construction of a model for a covariant Zeeman effect [26]. The solutions for the bound state problem in terms of the variables (5.9) are in the iirreducible representations of O(2,1). To achieve representations of O(3,1), the symmetry of the differential equations, we constructed an induced representation based on the coset space labelled by a spacelike vector n^μ. We then harmonically analyzed this induced representation over the O(3) subgroup of O(3,1), and found that in the scattering case, one could find the usual partial wave expansion if scattering occurred for the vector n^μ in the beam direction. In our study of the selection rules, we found that the vector

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n^μ suffered a recoil in the emission of electromagnetic radiation, and used this idea to construct a covariant Zeeman interaction.

We have studied, so far, applications in which strong agreement with known results was demonstrated. I conclude this section by citing two crucial experiments which can distinguish a theory in which the Einstein time is considered a dynamical variable from theories which do not study these properties of the time. The first is the possibility of *interference in time*, in exact analogy to the Davisson-Germer experiment for spatial interference from wave functions. Since the solutions of the equation (4.8) lie in a Hilbert space over R^4, they are coherent in time as well as space (as is well-known for electromagnetic waves); switching the flow off and on in time, by, for example the switching of a superconductor, has the effect of constructing a double slit in time. It is predicted that interference phenomena could be seen for electrons at frequencies of the order of 10^{12} Hz [27]. Sufficiently wide band amplifiers at this frequency are hard to construct, but perhaps coming into range. A second crucial experiment, no less difficult, involves measuring the differential cross section for scalar-scalar scattering very close to the forward direction. For off-shell particles, such cross sections should be asymptotically finite, and even show non-trivial structure (subsidiary peaks) in this region [28]. Studies are currently under way to investigate the feasibility of carrying out such experiments.

6. Comments and Conclusions

In addition to the studies reported above, some efforts have been made in developing a description of systems with spin. The Hermitian scalar product of the Hilbert space on R^4 precludes the use of finite dimensional representations of O(3,1) unless they are associated with an induced representation, of the type constructed by Wigner [9]. However, the necessity of computing expectation values of, say, the position x^μ, acting as a derivative in momentum space, rules out as well Wigner's use of the four-momentum to label the coset space. We therefore introduced a timelike four-vector [29][30] which commutes with all observables, for this purpose, and constructed a theory of systems with spin which resulted in a second order equation related to Dirac's, but with completely self-adjoint interaction between the spin and electromagnetic field. We have recently found a first order equation which iterates to that second order form, and requires, to achieve this, chirally projected wave functions [31].

The theory we have described above, which treat the time of Einstein as a dynamical variable, provides an integrable system of equations for the solution of problems of charges in interaction with electromagnetism. Consider, for example, two charged particles scattering through the electromagnetic force. The usual Maxwell formulation requires knowledge of one of the world lines in order to compute the electromagnetic field acting on the other everywhere in spacetime. One may then compute the trajectory of the other, and given this, the effect of its field on the first. The resulting motion of the first particle may then not be consistent with the original assumption, and the process of trial and error, or iteration, may be unstable. The computation of the five-potential pre-Maxwell fields, however, permits the stepwise integration of the coupled Stueckelberg and pre-Maxwell equations. The τ integration of the final result then determines the self-consistent Maxwell fields and currents [32]. Hence, raising the dimension of the structure of the theory provides an effective method for dealing with problems with back-reaction.

In a similar way, the problem of constructing gravitational systems with back-reaction, such as two stars in collision, may be facilitated by adding a dimension to the development of the Einstein theory. In this case, one generally must integrate the hydrodynamic model for the energy momentum tensor, the right hand side of the Einstein equations, to obtain a conserved energy-momentum tensor [15]. If we recognize that the transformation from the locally flat coordinates of a freely falling frame to the spacetime manifold of general relativity may depend on the universal world time, τ, then the derivatives of the local coordinates ξ^μ depend upon τ as well as the (curved) x^μ, i.e.,

$$d\xi^\mu = \frac{\partial \xi^\mu}{\partial x^\lambda} dx^\lambda + \frac{\partial \xi^\mu}{\partial \tau} d\tau;$$

substituting this formula into the expression for the invariant interval leads to a *five*-dimensional metric tensor, somewhat similar in form to that of Kaluza-Klein. The energy momentum tensor, before integration over its world history, would then form the source term for this (pre-)Einstein theory. Some preliminary investigations have been made of this structure [33].

Acknowledgements

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